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MINIMAL SYMMETRY, RANDOM AND DISORDER

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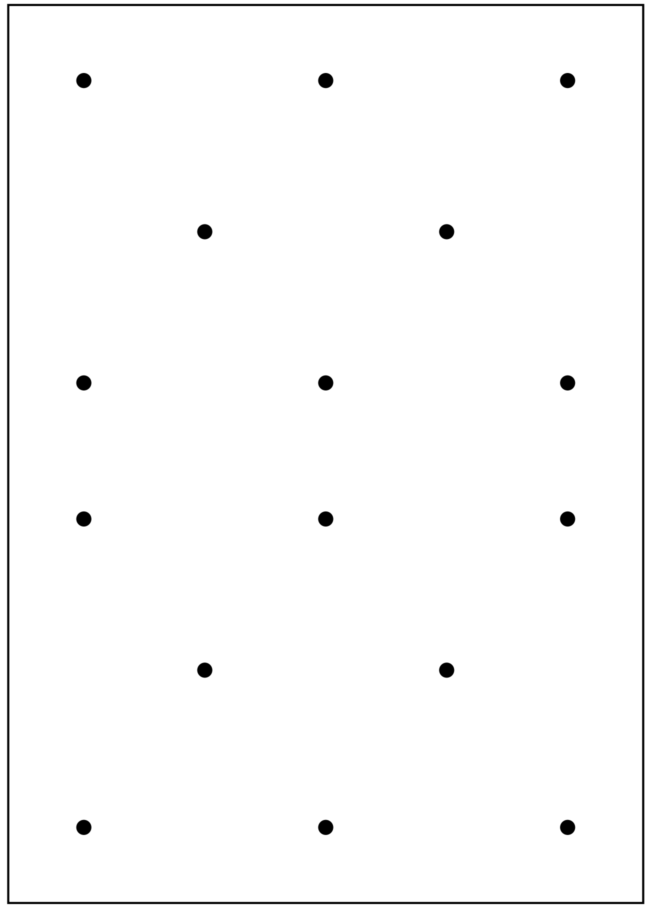
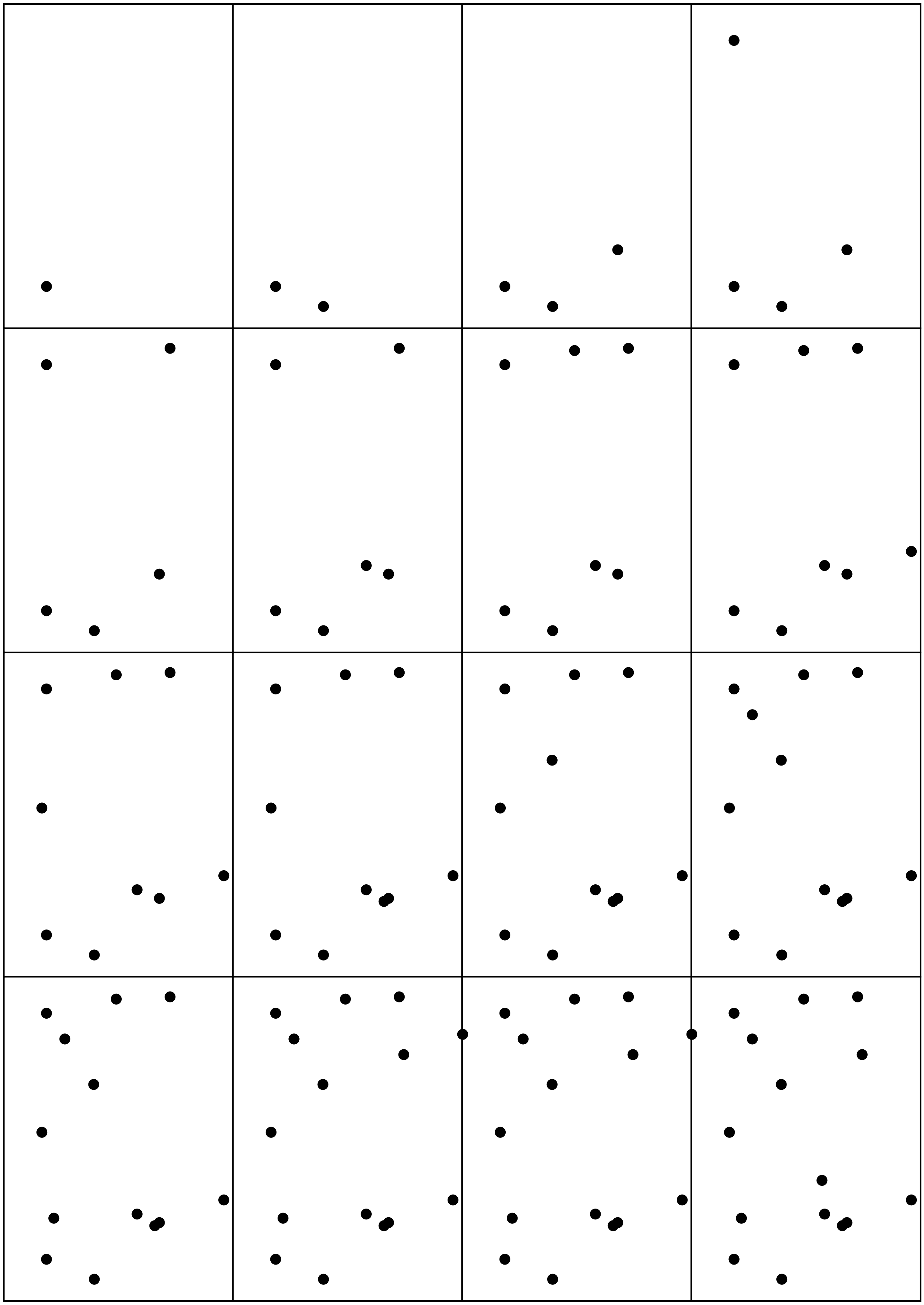


$$\text{Symmetry} \nearrow \qquad \Longleftrightarrow \qquad \text{Order} \nearrow$$

$$\text{RANDOM} \nearrow \qquad \Longrightarrow \qquad \text{symmetry ?} \qquad \text{Order ?}$$



RANDOM $\nearrow$ $\Longrightarrow$ SYMMETRY $\nearrow$ $\Longleftrightarrow$ ORDER $\nearrow$

MINIMAL SYMMETRY $\Longleftrightarrow$ MAXIMAL DISORDER

What is "MINIMAL SYMMETRY" ?

Need to measure QUANTITATIVELY symmetry

Example: the chiral index

HOW TO MEASURE SYMMETRY ?

(simplified)

(1) Define a space E of objects, and define when two objects are "equal"

(the equality is a relation of equivalence on E)

(2) Define a group operating on E

(Classifying symmetry relies on the group structure, not on E)

An object is symmetric if it is identical to one of its non-trivial transforms

(3) Define a "distance" between objects

(4) The minimized distance between an object and its non trivial transforms

is a measure of symmetry.

(this distance is normalized so that the measure is scale independant)

THE CHIRAL INDEX

(simplified)

(1) E: Space of probability distributions in the euclidean space

Set of points: sample

Colored points: mixture of colored distributions

When there is no color, we are just looking for a skewness measure

(2) Group structure: isometries with odd number of reflections

(3) Distance between distributions: Wasserstein L2

(is the generalization of the least square method: RMS, Procrustes,...)

Connection with the Monge-Kantorovitch transportation problem.

(4) Minimized distance between the object (distribution) and its inverted translated and rotated copies, normalized to the inertia.

At the difference of Pearson's skewness, a null chiral index implies that the distribution is indirect-symmetric

COMPUTATION OF THE CHIRAL INDEX

General: $\chi = \frac{\mathbf{d}}{\mathbf{4T}} \mathbf{Inf}_{\{\mathbf{W},\mathbf{R},\mathbf{t}\}} \mathbf{E}(\mathbf{X} - \tilde{\mathbf{X}})'(\mathbf{X} - \tilde{\mathbf{X}})$

For a serie of n observations, use a pocket calculator:

(1) Sort the n values in increasing order

(2) Correlate the serie with itself, sorted in reversed order

(3) Add 1, then divise by 2.

Example: 3 points $\chi = \frac{(\mathbf{1}-\alpha)^{\mathbf{2}}}{\mathbf{4}(\mathbf{1}+\alpha+\alpha^{\mathbf{2}})}$

THE RANDOM DISORDER PARADOX

Consequence of a convergence theorem:

The chiral index of the sample converges to

the chiral index of the paraent distribution

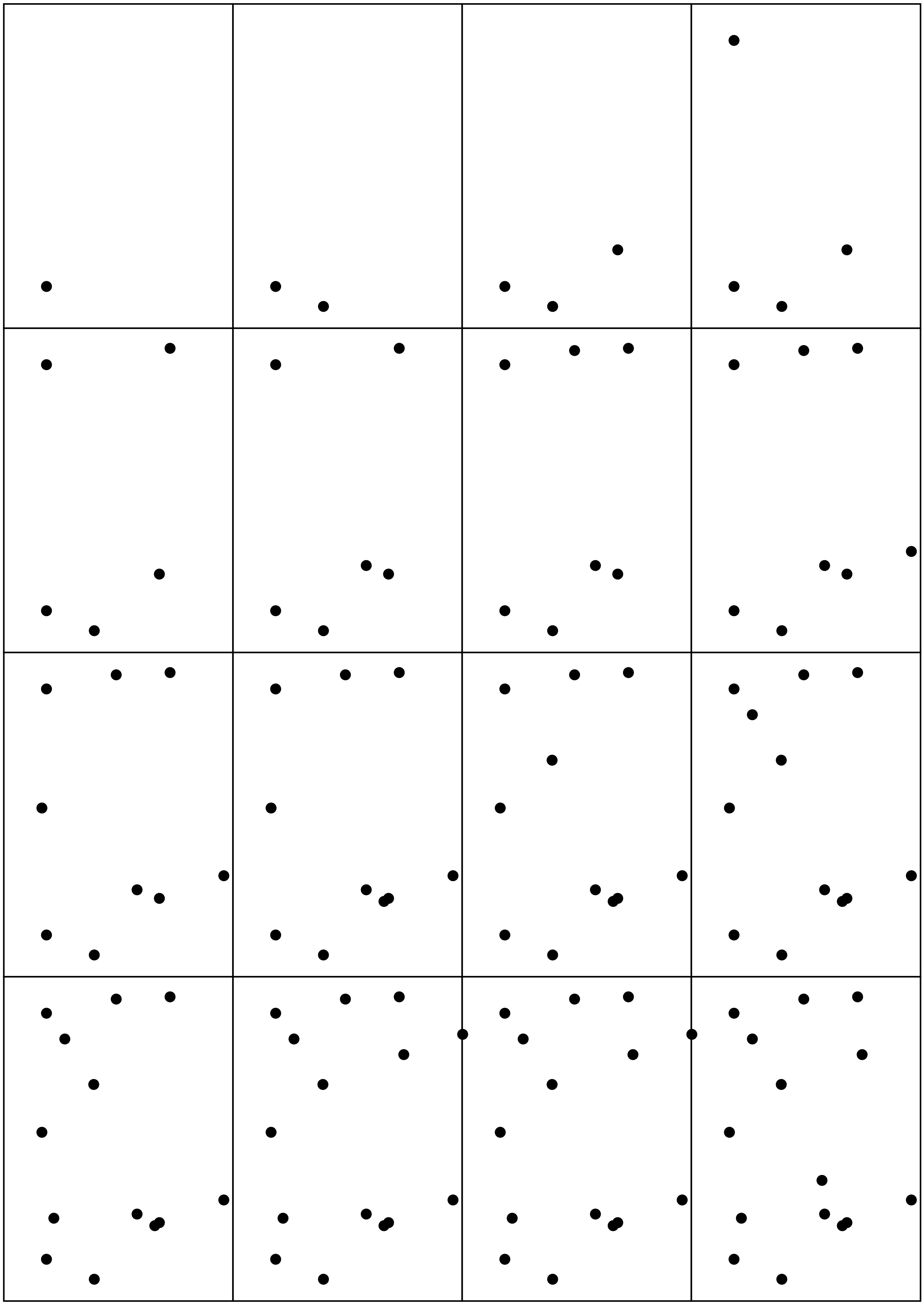
MORE RANDOM POINTS THERE IS, MORE SYMMETRY THERE IS

(indirect-symmetric parent distribution assumed)

MORE SYMMETRY THERE IS, MORE ORDER THERE IS

PARADOX: MORE RANDOM THERE IS, LESS DISORDER THERE IS

!!!



HOW INCREASE DISORDER ?

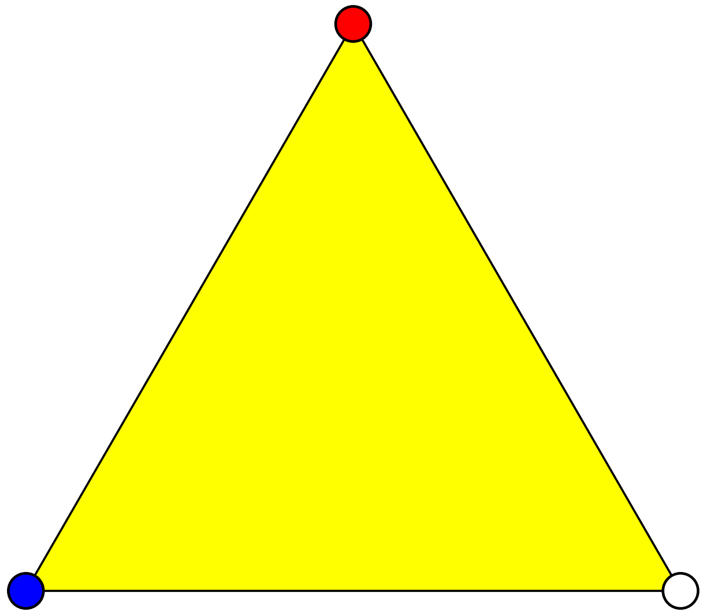
WE LOOK FOR MINIMAL SYMMETRY

NEEDS TO SOLVE A DIFFICULT OPTIMIZATION PROBLEM

SOLUTIONS KNOWN IN SOME PARTICULAR SITUATIONS

FOR DISTRIBUTIONS (NO COLOR), THE PROBLEM IS OPENED

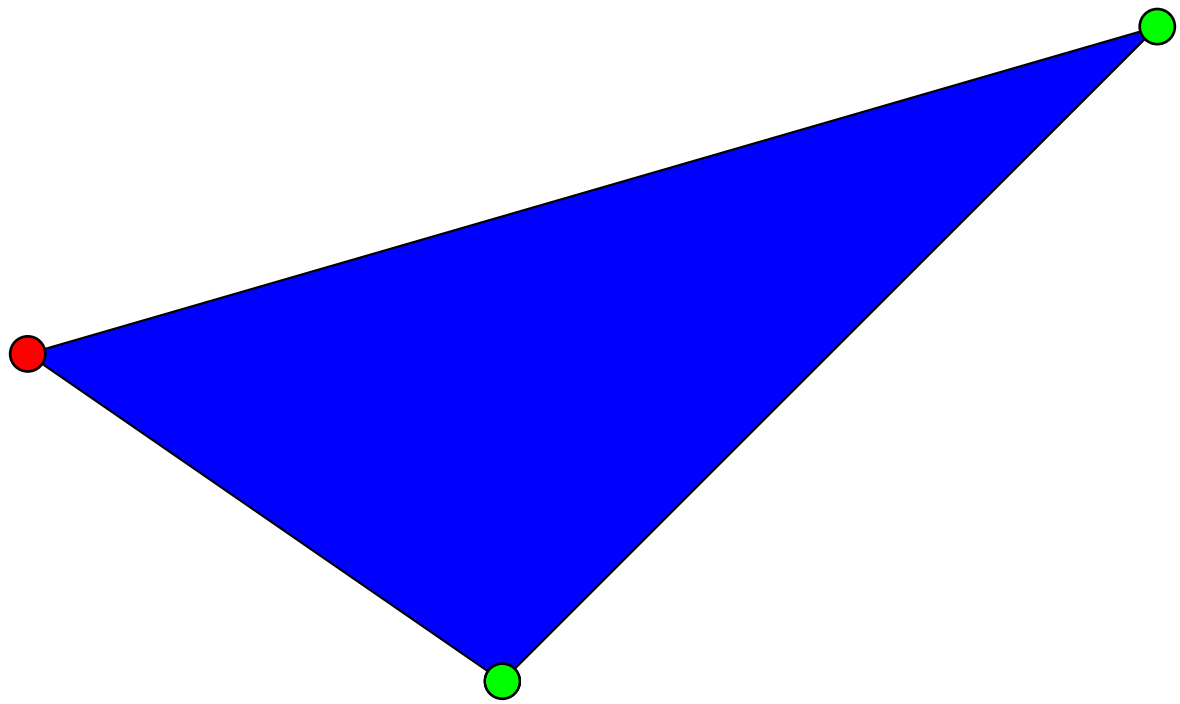
IN THE PLANE AND IN THE SPACE (AND IN HIGHER DIMENSIONS)



THE MOST CHIRAL TRIANGLE WITH ALL NON-EQUIVALENT VERTICES IS EQUILATERAL

CHI=1

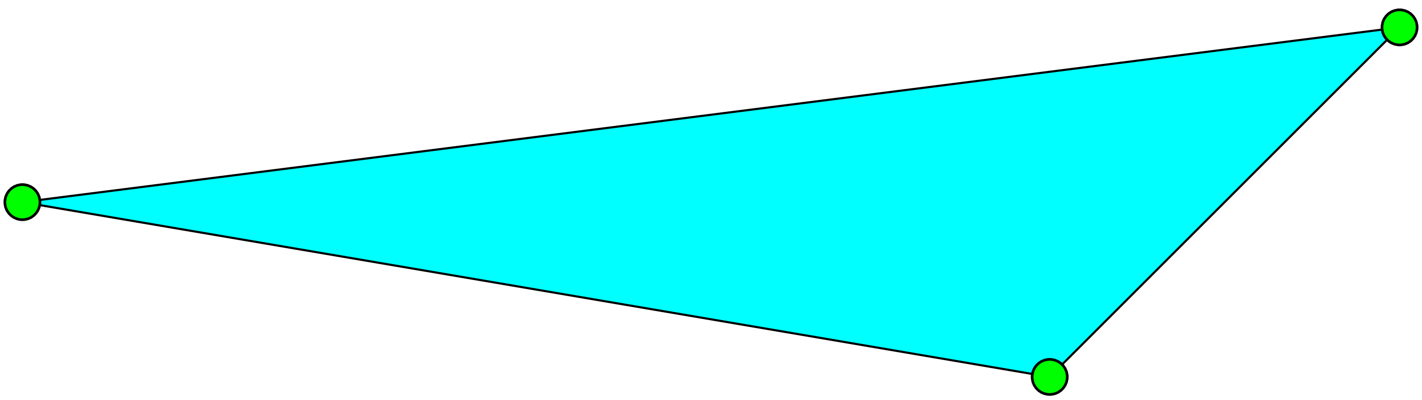
This result generalizes in any dimension: the most chiral simplex with all non-equivalent vertices is regular: CHI=1.



THE MOST CHIRAL TRIANGLE WITH 2 EQUIVALENT VERTICES

Distances ratio: $\sqrt{\mathbf{1}-\sqrt{\mathbf{6}}/\mathbf{4}}:\mathbf{1}:\sqrt{\mathbf{1}+\sqrt{\mathbf{6}}/\mathbf{4}}$

CHI=$\mathbf{1}-\sqrt{\mathbf{2}}/\mathbf{2}$



THE MOST CHIRAL TRIANGLE WITH 3 EQUIVALENT VERTICES

Distances ratio: $\mathbf{1} : \sqrt{\mathbf{4}+\sqrt{\mathbf{15}}} : \sqrt{(\mathbf{5}+\sqrt{\mathbf{15}})/\mathbf{2}}$

CHI=$\mathbf{1}-\mathbf{2}\sqrt{\mathbf{5}}/\mathbf{5}$

$\triangle$

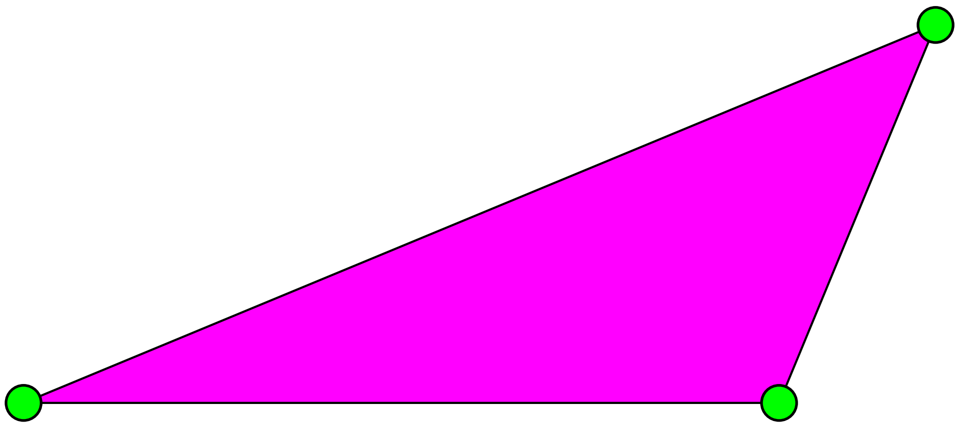
THE UNEQUIVALENCE OF ALL VERTICES PRECLUDES THE EXISTENCE OF ANY DIRECT SYMMETRY:

AT LEAST 2 POINTS SHOULD BE EQUIVALENT.

ONE OF THE MOST DISSYMETRIC TRIANGLES WITH 2 UNEQUIVALENT VERTICES

Abscissas: $(\mathbf{-1-\sqrt{3})/2, (-1+\sqrt{3})/2, 1}$

THIS DEGENERATE TRIANGLE IS SUCH THAT DSI=1 IN ANY DIMENSION.



THE MOST DISSYMETRIC TRIANGLE WITH 3 EQUIVALENT VERTICES

Angles: $\pi/\mathbf{4}, \pi/\mathbf{8}, \mathbf{5}\pi/\mathbf{8}$

DSI=$\mathbf{1} - \sqrt{\mathbf{2}}/\mathbf{2}$

REMARKABLE PROPERTY OF THE 5 EXTREMAL TRIANGLES

The 5 extremal triangles have all the following geometric property. The squared lengths of the sides are equal to three times the squared distances vertex-barycenter:

$d^2(p2,p3)=3d^2(p1,g)$

$d^2(p1,p2)=3d^2(p2,g)$

$d^2(p3,p1)=3d^2(p3,g)$

g being the barycenter of the points p1,p2,p3.

CARE:

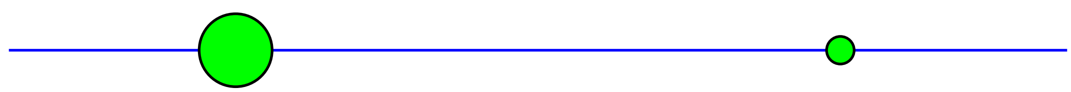
THE RELATION IS SYMMETRIC FOR TWO POINTS ONLY

SKEW DISTRIBUTIONS

ASYMPTOTIC MAXIMAL CHIRALITY FIGURES (

On the real line ($\mathbf{d = 1}$)

Bernouilli law of parameter tending to 0 or 1: Lim Sup (χ) $= \mathbf{1/2}$



In the plane ($\mathbf{d = 2}$)

Family of sets conjectured to be asymptotically of maximal chirality:

Lim Sup (χ) $= \mathbf{1 - 1}/\pi$

The calculations are easier in the complex plane

Fix $\epsilon > 0$ then choose even integer $m > 1/\epsilon$. $\omega = e^{i(2\pi)/(2m)}$ $(\omega^{2m} = 1)$

Select integer $r > m^4/\epsilon^2$ then select even integer $k > r^{m-1}/\epsilon$

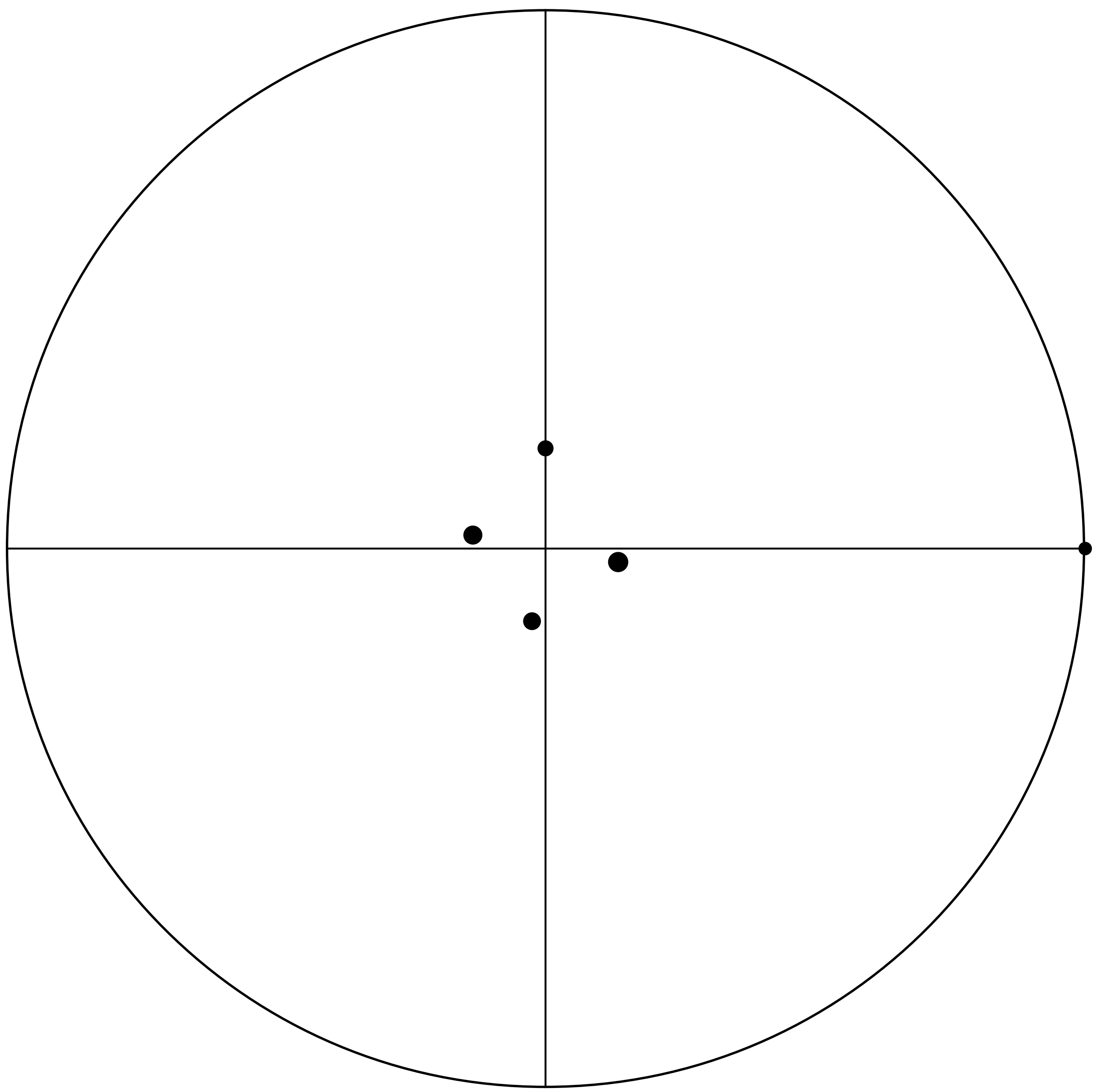
$z \in C^n$ z is a complex vector of $m + 3$ blocks of elements

Each block j, $j = 0..m+2$, contains identical elements.

$n = 1 + r + r^2 + \ldots + r^{m-1} + k + \frac{k}{2} + \frac{k}{2}$

$S = \sum_{j=0}^{j=m-1} \omega^j r^{j/2}$ (z is such that $z'\mathbf{1} = 0$ and $z'z = 0$)

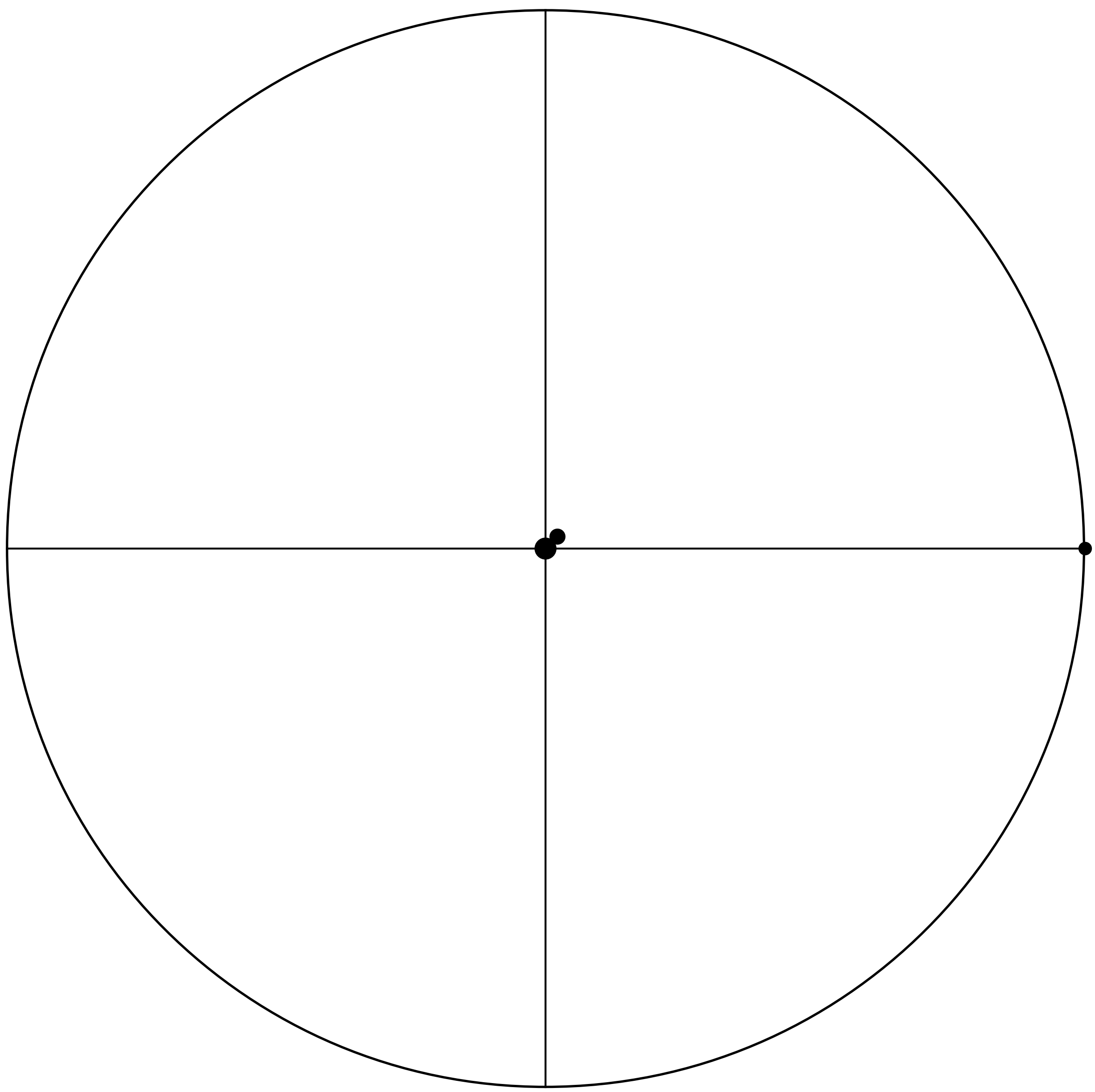
$block$	z_j	$multiplicity$
0	1	1
1	$\omega/r^{1/2}$	r
2	ω^2/r	r^2
$\vdots$	$\vdots$	$\vdots$
j	$\omega^j/r^{j/2}$	r^j
$\vdots$	$\vdots$	$\vdots$
$m-1$	$\omega^{m-1}/r^{(m-1)/2}$	r^{m-1}
m	$-S/k$	k
$m+1$	iS/k	$k/2$
$m+2$	$-iS/k$	$k/2$



$\epsilon = 0.750$

m = 2 ; m+3 = 5 ; r = 29

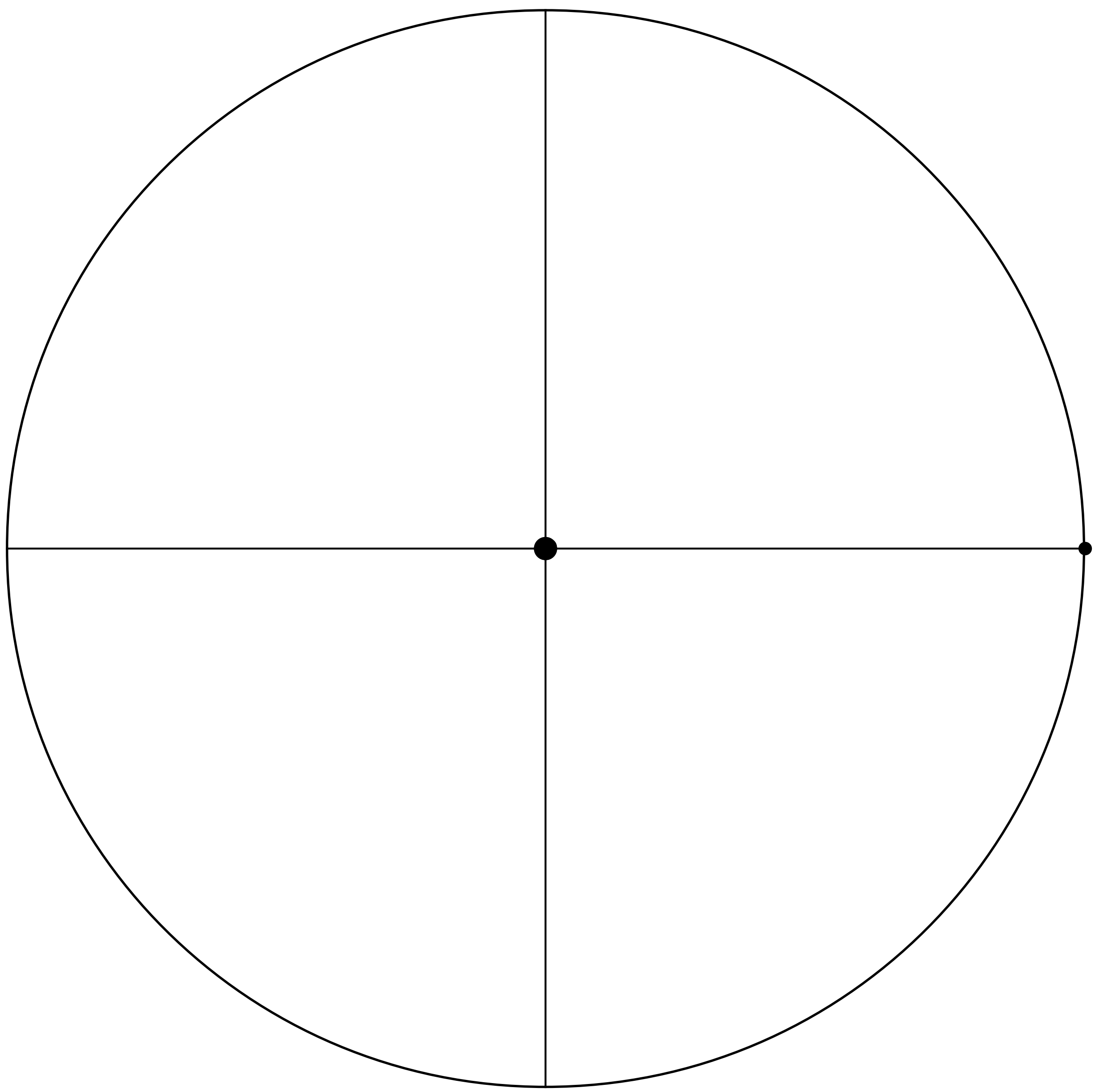
k = 0.400E+02 ; n = 0.110E+03



ϵ= 0.500

m = 4 ; m+3 = 7 ; r = 1025

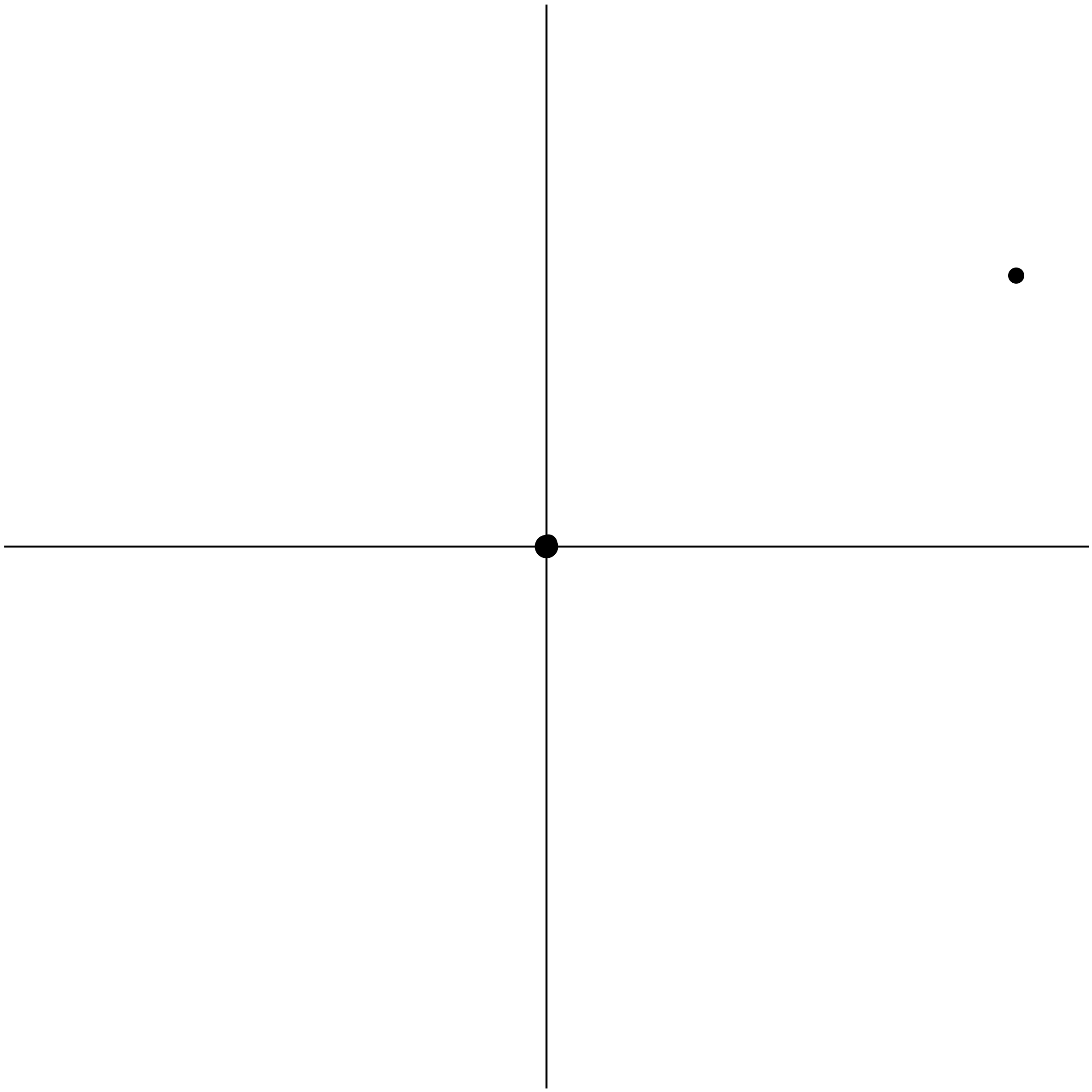
k = 0.215E+10 ; n = 0.539E+10



$\epsilon = 0.250$

m = 6 ; m+3 = 9 ; r = 20737

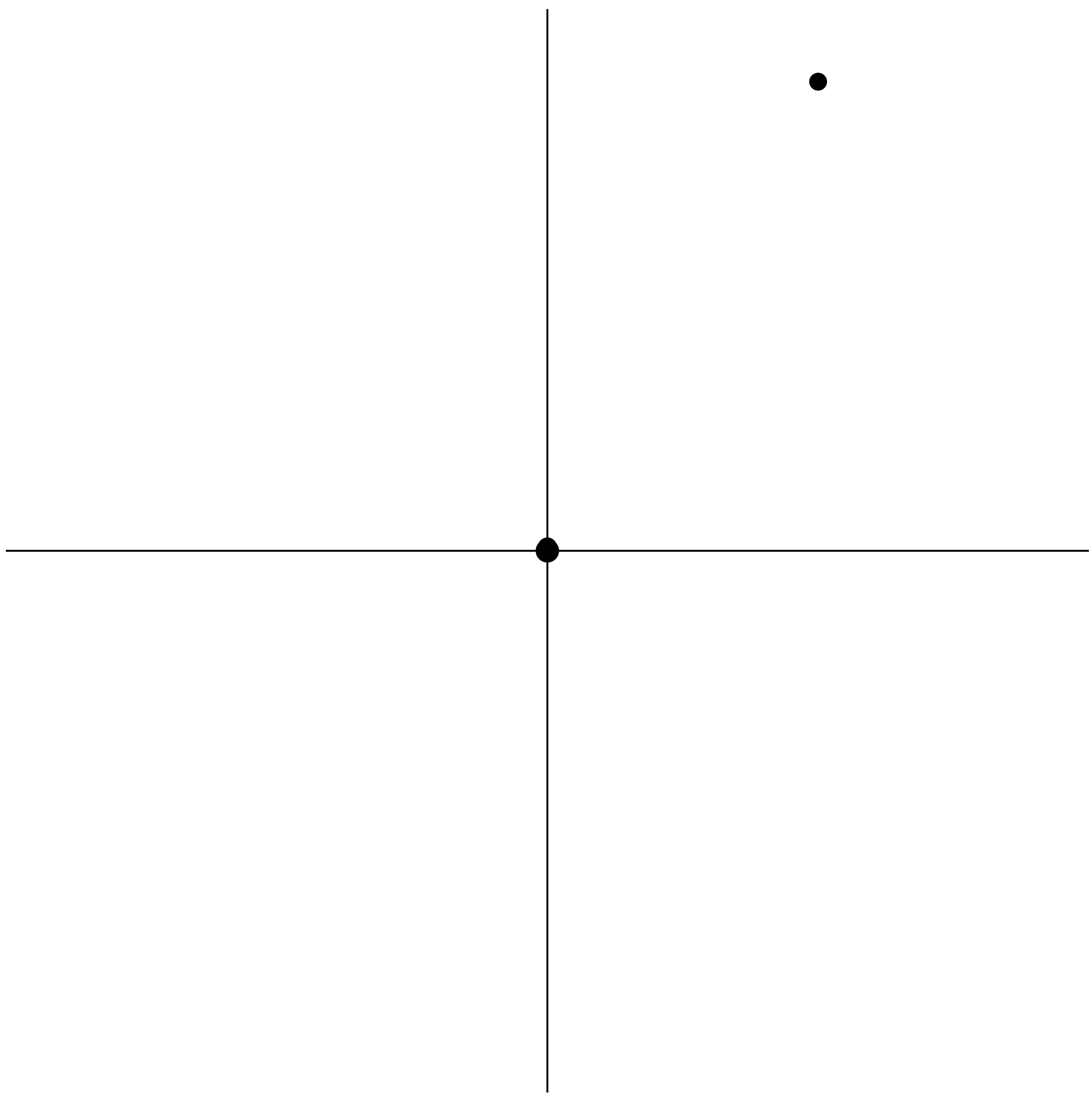
k = 0.153E+23 ; n = 0.345E+23



$\epsilon = 0.250$ Deleted points: 1 Scaling: 144.

m = 6 ; m+3 = 9 ; r = 20737

k = 0.153E+23 ; n = 0.345E+23



$\epsilon = 0.250$ Deleted points: 2 Scaling: 20737.

m = 6 ; m+3 = 9 ; r = 20737

k = 0.153E+23 ; n = 0.345E+23